Normal States and Unitary Equivalence of von Neumann Algebras.

R. V. KADISON

Mathematics Department, University of Pennsylvania - Philadelphia, Pa.

1. – Introduction.

These notes are the written version of lectures delivered at the University of Pennsylvania during March 1972. The lectures were one section of a graduate course on the theory of operator algebras attended by students well versed in the basics of the theory. It is a pleasure to record my gratitude to these students for their lively and stimulating participation: M. DUPRE, I. EBY, W. GREEN, D. LAISON, M. LANDSTAD, R. MCGOVERN, C. SKAU, G. STAMATOPOULOS and S. TSUI. Special thanks are due to D. LAISON for noticing that Lemma 2'3 as I presented it could be modified to apply to the noncommutative case. This allowed for a subtantial simplification in the presentation.

The guiding question is that of when two von Neumann algebras are unitarily equivalent. In essence the answer is contained in Theorem 3'7 (the unitary implementation theorem). With its aid and the techniques of restricting to invariant subspaces and tensoring, the question reduces completely to one of algebraic isomorphism. The main tool needed (as we present it) in making this reduction is a theory of normal states indicating when and how they may be represented as vector states.

2. – Vector state decomposition of completely additive states.

In this Section it is shown (Theorem 2'7) that completely additive states of von Neumann algebras are sums (possibly infinite) of positive vector functionals. The special feature of the argument is that it remains within the framework of the theory of operator algebras (that is, without a serious excursion into Banach-space theory).

2'1. *Definition.* – A completely additive (c.a.) state of a von Neumann algebra is a state ω of $\mathscr{R}$ such that $\omega\left(\sum_a E_a\right) = \sum_a \omega(E_a)$ for each orthogonal family $\{E_a\}$ of projections in $\mathscr{R}$. A carrier (support) for ω is a projection E such that $I-E=\sum_a E_a$ and $\{E_a\}$ is an orthogonal family of projections in $\mathscr{R}$ maximal with respect to the property that $\omega(E_a)=0$ for all a.

With the notation of the preceding definition, $\omega(I-E)=\omega\left(\sum_a E_a\right)=$ $=\sum_a \omega(E_a)=0$ and, since $I-E\geqslant 0$ and ω is a state, $0=\omega(T(I-E))=$ $=\omega((I-E)T)$ for each T in $\mathscr{R}$. If $A>0$, $A\in\mathscr{R}$ and $EAE=A$, then $\omega(A)>0$. In this case, $A\geqslant\lambda E_0$ for some nonzero spectral projection E_0 of A and some positive λ. Thus $E_0\leqslant E$; and $0<\omega(E_0)\leqslant\lambda^{-1}\omega(A)$ by maximality of $\{E_a\}$. We could argue to show that there is only one carrier for ω, at this point; but it is not needed and will be clear from later results.

2·2. *Lemma.* – If ω is a c.a. state of the von Neumann algebra $\mathscr{R}$, there is a countable orthogonal family of projections $\{E_n\}$ and vectors $\{x_n\}$ such that $\omega_{E_n}\leqslant\omega_{x_n}|\mathscr{R}$, $n=1,2,\ldots$, where $\omega\left(\sum_n E_n\right)=\sum_n \omega(E_n)=1$.

Proof. Let F be a carrier for ω. If we can find $\{E_n\}$ and $\{x_n'\}$, as in the statement of the Lemma, but for $F\mathscr{R}F$ and $\omega|F\mathscr{R}F$, then, with $x_n=Fx_n'$, $\omega_{E_n}\leqslant\omega_{x_n}|\mathscr{R}$ (for $\omega_{E_n}(A)=\omega(E_nFAFE_n)\leqslant(FAFx_n',x_n')=\omega_{x_n}(A)$ when A is a positive operator in $\mathscr{R}$). Thus we may assume that $\omega(A)>0$ for each positive A in $\mathscr{R}$.

Note, next, that if E is a projection in $\mathscr{R}$ and x is a vector such that $\|Ex\|=1$, then there is a nonzero subprojection E_0 of E in $\mathscr{R}$ such that $\omega_{E_0}\leqslant\omega_{E_0x}|\mathscr{R}$. To see this, let us observe, first, that if $\omega(E')\leqslant(E'x,x)$ for each subprojection E' in $\mathscr{R}$ of a projection E, then $\omega_E\leqslant\omega_{Ex}|\mathscr{R}$. In this case, with A a positive operator in $\mathscr{R}$, we can approximate EAE as closely as we wish, in norm, by linear combinations $\sum_{j=1}^n \lambda_j F_j$, where $\lambda_j\geqslant 0$ and $\{F_j\}$ are (mutually orthogonal spectral) projections (for EAE) in $\mathscr{R}$ contained in E. Then $\sum_{j=1}^n \lambda_j\omega(F_j)\leqslant\sum_{j=1}^n\lambda_j(F_jx,x)=$ $=\left(\left(\sum_{j=1}^n\lambda_jF_j\right)x,x\right)$; so that $\omega(EAE)\leqslant\omega_x(EAE)$.

Thus either we may choose E as E_0 or there is some subprojection F_0 of E in $\mathscr{R}$ such that $\omega(F_0)>(F_0x,x)$. Let $\{F_b\}$ be a maximal orthogonal family of such projections. Then $\omega\left(\sum_b F_b\right)=\sum_b\omega(F_b)>\sum_b(F_bx,x)=\left(\left(\sum_b F_b\right)x,x\right)$; so that $E_0\neq 0$ (and, hence, $\omega(E_0)\neq 0$), where $E_0=E-\sum_b F_b$. (If, on the contrary, $\omega(E_0)=0$, then $E_0=0$ and

$$1\geqslant\omega(E)=\omega\left(\sum_b F_b\right)=\sum_b\omega(F_b)>\left(\left(\sum_b F_b\right)x,x\right)=(Ex,x)=\|Ex\|^2=1\,.)$$

By maximality, $\omega(E')\leqslant(E'x,x)$ for each subprojection E' of E_0 in $\mathscr{R}$. From the earlier argument $\omega_{E_0}\leqslant\omega_{E_0x}|\mathscr{R}$.

Next, let $\{E_a\}$ be an orthogonal family of projections in $\mathscr{R}$ maximal with respect to the property that there are vectors $\{x_a\}$ such that $\omega_{E_a} \leqslant \omega_{x_a}|\mathscr{R}$. Since $\sum_a \omega(E_a) = \omega\left(\sum_a E_a\right) \leqslant 1$ and $\omega(E_a) > 0$ unless $E_a = 0$, $\{E_a\}$ is a countable family, and we relabel it as $\{E_n\}$. If $0 \neq I - \sum_n E_n (= E)$, there is a unit vector x in the range of E; and, from our earlier argument, there is a subprojection E_0 of E in $\mathscr{R}$ such that $0 < \omega_{E_0} \leqslant \omega_{x_0}|\mathscr{R}$, where $x_0 = E_0 x$. But E_0 is orthogonal to each E_n contradicting the maximality of $\{E_n\}$. Thus $I = \sum E_n$ and $\sum \omega(E_n) = 1$.

2·3. *Lemma.* – If ω is a c.a. state of the von Neumann algebra $\mathscr{R}$ acting on $\mathscr{H}$, then ω is weak-operator continuous on $\mathscr{R}_1$, the unit ball of $\mathscr{R}$.

Proof. Let $\{E_n\}$ and $\{x_n\}$ be as in the preceding lemma. With $\varepsilon > 0$ given, find N such that $\omega\left(\sum_{n>N} E_n\right) < \varepsilon^2/64$. Let A be an operator in $\mathscr{R}_2$, the ball of radius 2 with center 0 in $\mathscr{R}$, such that $\|Ax_n\| < \varepsilon/4N$, $n = 1, \ldots, N$. Then, writing H for A^*A,

$$|\omega(A)|^2 \leqslant \omega(H) = \omega\left(H\sum_{n=1}^{N} E_n\right) + \omega\left(H\sum_{n>N} E_n\right) \leqslant$$

$$\leqslant \sum_{n=1}^{N} \omega(H^{\frac12}H^{\frac12}E_n) + \omega(H^2)^{\frac12}\omega\left(\sum_{n>N} E_n\right)^{\frac12} \leqslant \sum_{n=1}^{N} \omega(H)^{\frac12}\omega(E_nHE_n)^{\frac12} + \frac{\varepsilon}{2} \leqslant$$

$$\leqslant 2\sum_{n=1}^{N} \|Ax_n\| + \frac{\varepsilon}{2} \leqslant \varepsilon .$$

Thus ω is strong-operator continuous at 0 on $\mathscr{R}_2$. Since translation by an operator in $\mathscr{R}_1$ is a strong-operator continuous mapping of $\mathscr{R}_1$ into $\mathscr{R}_2$, ω is strong-operator continuous on $\mathscr{R}_1$. The closed half-planes in the plane of complex numbers are convex and have complements which form a subbase for the open sets. The inverse image of a closed half-plane intersects $\mathscr{R}_1$ in a convex set which is strong, hence, weak-operator closed. Its complement has inverse image which intersects $\mathscr{R}_1$ in a weak-operator open set. Thus ω is weak-operator continuous on $\mathscr{R}_1$.

2·4. *Corollary.* – If ω is a c.a. state of the von Neumann algebra $\mathscr{R}$, ω_B is weak-operator continuous on $\mathscr{R}_1$ (hence ω is a c.a. state when $\omega(B^*B) = 1$).

2·5. *Lemma.* – If $A \to \bar{A}$ is the direct sum of all representations of a von Neumann algebra $\mathscr{R}$ engendered by c.a. states of $\mathscr{R}$, then $\bar{\mathscr{R}}$ is weak-operator closed.

Proof. If $\bar{\mathscr{R}}$ acts on $\mathscr{H}$, ω is a c.a. state of $\mathscr{R}$, and x is a unit vector in $\mathscr{H}$ such that $\omega(A) = (\bar{A}x, x)$ for all A in $\mathscr{R}$, then $\omega_B(A) = (\bar{A}\bar{B}x, \bar{B}x)$. Thus $A \to \to (\bar{A}\bar{B}x, \bar{B}x)$ is weak-operator continuous on $\mathscr{R}_1$. With $\{\omega_1, \ldots, \omega_n\}$ a finite set of c.a. states of $\mathscr{R}$ and $x_1, \ldots, x_n$ the corresponding vectors in $\mathscr{H}$,

$\left\{\sum_{j=1}^{n}\bar{B}_j x_j : B_j \text{ in } \mathscr{R}\right\}$ is dense in $\mathscr{H}$. Moreover,

$$A \to \left(\bar{A}\sum_{j=1}^{n}\bar{B}_j x_j,\ \sum_{j=1}^{n}\bar{B}_j x_j\right) = \sum_{j=1}^{n}(\bar{A}\bar{B}_j x_j,\ \bar{B}_j x_j)$$

is weak-operator continuous on $\mathscr{R}_1$. For each z in $\mathscr{H}$, then, $A \to (\bar{A}z, z)$ is a uniform limit of weak-operator continuous functions on $\mathscr{R}_1$ and is weak-operator continuous on $\mathscr{R}_1$. Thus $A \to \bar{A}$ is continuous from $\mathscr{R}_1$ in its weak-operator topology to $\bar{\mathscr{R}}_1$ in its weak-operator topology. But $\mathscr{R}_1$ is weak-operator compact; so that $\bar{\mathscr{R}}_1$ is weak-operator compact (hence, closed). The Kaplansky density theorem implies that $\bar{\mathscr{R}}$ is weak-operator closed.

2·6. *Corollary.* – If ω is a c.a. state of the von Neumann algebra $\mathscr{R}$ and φ the representation of $\mathscr{R}$ it engenders, then $\varphi(\mathscr{R})$ is weak-operator closed.

Proof. If x is a vector in $\mathscr{H}$ such that $\omega(A) = (\bar{A}x, x)$, in the notation of the preceding lemma, then $A \to \bar{A}E'$ is unitarilye quivalent to φ, where E' has range $[\bar{\mathscr{R}}x]$. But $\bar{\mathscr{R}}E'$ is weak-operator closed.

2·7. *Theorem.* – If ω is a c.a. state of a von Neumann algebra $\mathscr{R}$ acting on the Hilbert space $\mathscr{H}_0$, there is a countable set of vectors $\{x_n\}$ in $\mathscr{H}_0$ such that $\omega = \sum_{n=1}^{\infty}\omega_{x_n}|\mathscr{R}\ \left(\text{and } \sum_{n=1}^{\infty}\|x_n\|^2 = 1\right)$.

Proof. Let $\bar{\mathscr{R}}$ and $\mathscr{H}$ be as in Lemma **2·5**. Let $\{E_a'\}$ be a maximal orthogonal family of cyclic projections in $\mathscr{R}'$ with orthogonal central carriers $\{C_{E_a'}\}$. If ω_a is the vector state of $\mathscr{R}$ corresponding to a unit cyclic vector for E_a' (under $\mathscr{R}$) and x_a is a unit vector in $\mathscr{H}$ such that $\omega_a(A) = (\bar{A}x_a, x_a)$ for all A in $\mathscr{R}$, then $A \to AE$ is unitarily equivalent to $A \to \bar{A}\bar{E}_a'$, where $\bar{E}_a'$ is the (cyclic) projection in $\bar{\mathscr{R}}'$ with range $[\bar{\mathscr{R}}x_a]$. Thus, writing E' for $\sum_a E_a'$, the isomorphism $A \to AE'$ is unitarily equivalent to $A \to \bar{A}\bar{E}_a'$, where $\bar{E}' = \sum_a \bar{E}_a'$. Note, for this, that $C_{E'} = \sum_a C_{E_a'} = I$, by maxmality of $\{E_{a'}\}$. Hence $C_{\bar{E}'} = I$. From the Comparison Theorem, there is a family of projections $\{\bar{E}_{a,b,c}'\}$ with sum I such that $\{\bar{E}_{a,b,c}'\}_c$ are all equivalent to a subprojection $\bar{F}_{a,b}'$ of $\bar{E}_a'$. Let $V_{a,b,c}'$ be a partial isometry in $\bar{\mathscr{R}}'$ with initial space $\bar{E}_{a,b,c}'$ and final space $\bar{F}_{a,b}'$. If y is a unit vector in $\mathscr{H}$ such that $\omega(A) = (\bar{A}y, y)$ for all A in $\mathscr{R}$, then at most a countable family of $V_{a,b,c}'\bar{E}_{a,b,c}'y$ is nonzero. Denoting these as $y_1, y_2, \ldots$, note that $y_n \in \bar{E}'(\mathscr{H})$ and

$$\sum_n(\bar{A}y_n, y_n) = \sum_{a,b,c}(\bar{A}V_{a,b,c}'\bar{E}_{a,b,c}'y,\ V_{a,b,c}'\bar{E}_{a,b,c}'y) = \sum_{a,b,c}(\bar{A}\bar{E}_{a,b,c}'y,\ \bar{E}_{a,b,c}'y) =$$

$$= \left(\bar{A}\sum_{a,b,c}\bar{E}_{a,b,c}'y,\ \sum_{a,b,c}\bar{E}_{a,b,c}'y\right) = (\bar{A}y, y) = \omega(A)\,.$$

As $A \to AE'$ is unitarily equivalent to $A \to \bar{A}\bar{E}'$, there are vectors $\{x_n\}$ in $E'(\mathscr{H}_0)$ such that $\omega(A) = \sum_n (Ax_n, x_n) = \sum_n \omega_{\chi_n}(A)$. Since $1 = \omega(I)$, $\sum_n \|x_n\|^2 = 1$.

2˙8. *Corollary.* – If ω is a c.a. state of the von Neumann algebra $\mathscr{R}$ acting on the Hilbert space $\mathscr{H}$, there is an orthogonal family $\{y_n\}$ of vectors in $\mathscr{H}$ such that $\omega = \sum \omega_{y_n}|\mathscr{R}$ and $\sum \|y_n\|^2 = 1$.

Proof. From Theorem 2˙7 there is a family $\{x_n\}$ of vectors such that $\sum \omega_{x_n}|\mathscr{R} = \omega$ and $\sum \|x_n\|^2 = 1$. Then T is a self-adjoint trace class operator (positive, of trace 1), where $T = \sum \|x_n\|^2 E_n$ and E_n is the one-dimensional projection with χ_n in its range. It follows that

$$\omega(A) = \sum (Ax_n, x_n) = \sum \|x_n\|^2 \operatorname{Tr}(E_n A E_n) = \sum \|x_n\|^2 \operatorname{Tr}(E_n A) = \operatorname{Tr}(TA).$$

If $\{y'_n\}$ is an orthonormal basis of eigenvectors for T corresponding to the eigenvalues $\{a_n^2\}$ (with $a_n \geqslant 0$), then $\sum a_n^2 = 1$. With $y_n = a_n y'_n$

$$\omega(A) = \operatorname{Tr}(TA) = \operatorname{Tr}\left(\left(\sum a_n^2 F_n\right) A\right) = \sum a_n^2 \operatorname{Tr}(F_n A F_n) = \sum (Ay_n, y_n) = \sum \omega_{y_n}(A),$$

where F_n is the one-dimensional projection with y'_n in its range. Thus $\omega = \sum \omega_{y_n}|\mathscr{R}$.

2˙9. *Definition.* – The ultra-weak topology on a von Neumann algebra $\mathscr{R}$ acting on a Hilbert space $\mathscr{H}$ is the weakest topology relative to which all functionals of the form $\sum_{n=1}^{\infty} \omega_{x_n, y_n}|\mathscr{R}$, with $\sum (\|x_n\|^2 + \|y_n\|^2) < \infty$, are continuous.

Remarks. From Phillips' theorem, each functional on $\mathscr{R}$ continuous in the ultraweak topology has the form $\sum_{n=1}^{\infty} \omega_{x_n, y_n}|\mathscr{R}$, with $\sum (\|x_n\|^2 + \|y_n\|^2) < \infty$ (since these form a linear subspace of the dual of $\mathscr{R}$). Note that

$$\left|\sum \omega_{x_n, y_n}(A)\right| = \left|\sum (Ax_n, y_n)\right| \leqslant \sum \|A\| \, \|x_n\| \, \|y_n\| \leqslant \|A\| \left(\sum \|x_n\|^2\right)^{\frac{1}{2}} \left(\sum \|y_n\|^2\right)^{\frac{1}{2}};$$

so that, under the hypothesis $\sum (\|x_n\|^2 + \|y_n\|^2) < \infty$, $\sum_{n=1}^{\infty} \omega_{x_n, y_n}(A)$ converges (absolutely) and $\sum \omega_{x_n, y_n}|\mathscr{R}$ is in the norm-dual of $\mathscr{R}$.

2˙10. *Lemma.* – An ultra-weakly continuous linear functional ω on a von Neumann algebra $\mathscr{R}$ is weak-operator continuous on the unit ball $\mathscr{R}_1$, of $\mathscr{R}$.

Proof. From the preceding remark, $\omega = \sum_{n=1}^{\infty} \omega_{x_n, y_n}|\mathscr{R}$, where $\sum_{n=1}^{\infty} (\|x_n\|^2 + \|y_n\|^2) < \infty$. Thus $\|\omega - \omega_k\| \underset{k}{\to} 0$, where $\omega_k = \sum_{n=1}^{k} \omega_{x_n, y_n}|\mathscr{R}$. As each ω_k is weak-operator continuous on $\mathscr{R}_1$, the same is true of ω.

2·11. *Definition.* – A state ω of a von Neumann algebra $\mathcal{R}$ is said to be normal when $\omega(A_a)\uparrow\omega(A)$ for each monotone increasing net of operators (A_a) in $\mathcal{R}$ with least upper bound A.

2·12. *Theorem.* – The following conditions on a state ω of a von Neumann algebra $\mathcal{R}$ acting on $\mathcal{H}$ are equivalent:

a) $\omega = \sum_{n=1}^{\infty} \omega_{y_n}|\mathcal{R}$ and $\sum \|y_n\|^2 = 1$, with $\{y_n\}$ an orthogonal family of vectors in $\mathcal{H}$;

b) $\omega = \sum_{n=1}^{\infty} \omega_{x_n}|\mathcal{R}$ and $\sum \|x_n\|^2 = 1$;

c) ω is ultra-weakly continuous on $\mathcal{R}$;

d) ω is weakly continuous on the unit ball of $\mathcal{R}$;

e) ω is strongly continuous on the unit ball of $\mathcal{R}$;

f) ω is normal on $\mathcal{R}$;

g) ω is a c.a. state of $\mathcal{R}$.

Proof. The implications $a) \to b) \to c)$ are immediate. Lemma **2·10** yields $c) \to d)$. Since the weak-operator topology is weaker than the strong-operator topology, $d) \to e)$ follows. If $A_a \uparrow A$, the same is true for the (bounded) co-final subnet of (A_a) consisting of those A_a greater than or equal to a fixed A_{a_0}. As this convergence is in the strong-operator topology $\omega(A_a)$ converges to $\omega(A)$ over this subnet, under the assumption of $e)$. Thus $\omega(A_a)\uparrow\omega(A)$. If $E = \sum E_a$ for an orthogonal family $\{E_a\}$ of projections in $\mathcal{R}$, then $\left(\sum_{a \text{ in } \mathcal{F}} E_a\right)$ is a monotone increasing net of projections in $\mathcal{R}$ indexed by finite subsets $\mathcal{F}$ of the family $\{a\}$. Thus $\sum_{a \text{ in } \mathcal{F}} \omega(E_a)\uparrow\omega(E)$, under the assumption $f)$; so that $\omega(E) = \sum_a \omega(E_a)$. Hence $f) \to g)$. Finally, $g) \to a)$ is precisely the assertion of Corollary **2·8**.

3. – When normal states are vector states. The unitary implementation theorem.

The goal of this Section is Theorem **3·3**, in the presence of a separating vector each normal (c.a.) state is a vector state. With its aid, we prove the unitary implementation theorem (Theorem **3·7**). Further use of it results in the comparison theorem for pairs of jointly generated cyclic projections in a von Neumann algebra and its commutant (Theorem **3·9**).

3·1. *Lemma.* – If $\mathcal{R}$ is a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and x_0 is a unit cyclic vector for $\mathcal{R}$, then, for each vector z_0 in $\mathcal{H}$, there are operators, T, S in $\mathcal{R}$ and a vector y_0 orthogonal to the null space of T such that $Ty_0 = x_0$ and $Sy_0 = z_0$.

Proof. Assume $\|z_0\| = 1$. Since x_0 is cyclic for $\mathcal{R}$, there are operators T_n in $\mathcal{R}$ such that $\sum_{n=0}^{\infty} T_n x_0 = z_0$ and $\|T_n x_0\| \leqslant 4^{-n}$. If $H_n^2 = I + \sum_{k=0}^{n} 4^k T_k^* T_k$, then (H_n^2) is a monotonic increasing sequence of positive invertible operators in $\mathcal{R}$; and (H_n^{-2}) is a monotonic decreasing sequence of positive invertible operators in $\mathcal{R}$. Thus (H_n^{-2}) is a bounded sequence of operators tending strongly to some operator in $\mathcal{R}$ and H_n^{-1} $(= (H_n^{-2})^{\frac{1}{2}})$ tends strongly to an operator T in $\mathcal{R}$.

Note that

$$\|H_n x_0\|^2 = (H_n^2 x_0, x_0) = \left(\left(I + \sum_{k=0}^{n} 4^k T_k^* T_k\right) x_0, x_0\right) =$$

$$= (x_0, x_0) + \sum_{k=0}^{\infty} 4^k \|T_k x_0\|^2 \leqslant 1 + \sum_{k=0}^{\infty} 4^{-k} < 3 .$$

Since the ball of radius r is weakly compact in $\mathcal{H}$, some subsequence of $(H_n x_0)$, say $(H_{n'} x_0)$, converges weakly to y_0 in $\mathcal{H}$. We assert that $Ty_0 = x_0$. If $\varepsilon > 0$ and x is a vector in $\mathcal{H}$, there is a positive integer N such that, if $n \geqslant N$ and $n' \geqslant N$, then $\|(H_n^{-1} - T)x\| < \varepsilon/6$ and $|(H_{n'} x_0 - y_0, Tx)| < \varepsilon/2$. In this case, with $n' \geqslant N$

$$|(H_{n'} x_0, Tx) - (H_{n'} x_0, H_n^{-1}, x)| = |(H_{n'} x_0, Tx) - (x_0, x)| < \frac{\varepsilon}{6} \|H_n, x_0\| < \frac{\varepsilon}{2} ,$$

so that $|(y_0, Tx) - (x_0, x)| = |(Ty_0 - x_0, x)| < \varepsilon$ and $Ty_0 = x_0$ as asserted.

With m larger than n

$$0 \leqslant H_m^{-1} 4^n T_n^* T_n H_m^{-1} \leqslant H_m^{-1} \left(\sum_{k=0}^{m} 4^k T_k^* T_k\right) H_m^{-1} \leqslant I .$$

Now $H_m^{-1} 4^n T_n^* T_n H_m^{-1}$ tends strongly to $4^n TT_n^* T_n T$ as m tends to ∞; so that $0 \leqslant 4^n TT_n^* T_n T \leqslant I$, and $\|T_n T\| \leqslant 2^{-n}$. Thus $\sum_{n=0}^{\infty} T_n T$ converges, in norm, to an operator S in $\mathcal{R}$. Since both T and S have the same effect on y_0 as they do on its component orthogonal to the null space of T, we may assume y_0 is orthogonal to that null space. As $Ty_0 = x_0$, $Sy_0 = \sum_{n=0}^{\infty} T_n Ty_0 = \sum_{n=0}^{\infty} T_n x_0 = z_0$.

3·2. *Corollary.* – If $\mathcal{R}$ is a von Neumann algebra acting on $\mathcal{H}$, F' is the projection in $\mathcal{R}'$ with range $[\mathcal{R}x_0]$, where x_0 is a separating vector for $\mathcal{R}$, then $E' \leqslant F'$, for each cyclic projection E' in $\mathcal{R}'$.

Proof. Since x_0 is cyclic for $\mathscr{R}'$, with z_0 a generating vector for the range of E', there are operators T', S' in $\mathscr{R}'$ and a vector y_0 orthogonal to the null space of T' such that $T'y_0 = x_0$ and $S'y_0 = z_0$. If M' is the projection in $\mathscr{R}'$ with range $[\mathscr{R}y_0]$, then $E' \leqslant M'$; for $[\mathscr{R}z_0] = [S'\mathscr{R}y_0] = [\text{range } S'M']$. But $r(S'M') \sim$ $\sim r(M'S'^*) \leqslant M'$. Similarly, $F' \leqslant M'$, and $F' \sim r(M'T'^*)$. As $y_0 \in [\text{range } T'^*]$ and $M'y_0 = y_0$; $r(M'T'^*)y_0 = y_0$. Thus $M' \leqslant r(M'T'^*)$; and $F' \gtrsim M'$. Hence $E' \lesssim F'$.

3˙3. *Theorem.* – If ω is a c.a. state of the von Neumann algebra $\mathscr{R}$ acting on $\mathscr{H}_0$ and x_0 is a separating vector for $\mathscr{R}$, then there is a vector y_0 in $\mathscr{H}_0$ such that $\omega = \omega_{y_0}|\mathscr{R}$.

Proof. If $\bar{\mathscr{R}}$ acting on $\mathscr{H}$ is the « universal normal » representation of $\mathscr{R}$ (described in Lemma **2**˙5), there are vectors x and y in $\mathscr{H}$ such that $\omega(A) =$ $= (\bar{A}y, y)$ and $\omega_{x_0}(A) = (\bar{A}x, x)$, for each A in $\mathscr{R}$. In addition, if F' is the projection in $\bar{\mathscr{R}}'$ with range $[\bar{\mathscr{R}}x]$, the mapping $Ax_0 \to \bar{A}x$ extends to a unitary transformation U of $[Rx_0]$ onto $F'(\mathscr{H})$ such that $UAU^* = \bar{A}F'$. Since x_0 is separating for $\mathscr{R}$, x is separating for $\bar{\mathscr{R}}$; and $E' \lesssim F'$, where $E'(\mathscr{H}) = [\bar{\mathscr{R}}y]$. If V' is a partial isometry in $\bar{\mathscr{R}}'$ such that $V'^*V' = E'$ and $V'V'^* \leqslant F'$, then $\omega = \omega_{y_0}|\mathscr{R}$, where $y_0 = U^*V'y$.

3˙4. *Definition.* – The *carrier* (or, *support*) of a normal state of a von Neumann algebra is the orthogonal complement of the union of all projections annihilated by the state.

Remark A. If the support of ω on $\mathscr{R}$ is E, then $\omega(I - E) = 0$; for if A in $\mathscr{R}$ is positive and $\omega(A) = 0$, then $0 \leqslant \omega(A^n) = \omega(A^{n-\frac{1}{2}}A^{\frac{1}{2}}) \leqslant \omega(A^{2n-1})^{\frac{1}{2}}\omega(A)^{\frac{1}{2}} = 0$. Thus $\omega(f(A))$ for each continuous function f on the spectrum of A such that $f(0) = 0$. In particular $\omega(A^{1/n}) = 0$; and, since ω is normal, ω annihilates the range projection, F, of A. Conversely if $\omega(F) = 0$, then $0 \leqslant \omega(A) = \omega(FA) \leqslant$ $\leqslant \omega(F)^{\frac{1}{2}}\omega(A^2)^{\frac{1}{2}} = 0$. Thus two normal states have the same carrier if and only if they annihilate the same positive operators. If $\omega(M) = \omega(N) = 0$, then $\omega(M + N) = 0$; so that ω annihilates the range projection of $M + N$, that is, $\omega(M \vee N) = 0$. Since ω is normal, ω annihilates the union $I - E$ of all projections on which it vanishes.

Remark B. The support of $\omega_z|\mathscr{R}$ has range $[\mathscr{R}'z]$.

3˙5. *Lemma.* – If $\mathscr{R}$ is a von Neumann algebra acting on $\mathscr{H}$ and ω is a normal state of $\mathscr{R}$ with carrier E contained in the carrier of $\omega_z|\mathscr{R}$, then $\omega = \omega_y|\mathscr{R}$ with y in $[\mathscr{R}'z]$.

Proof. The carrier F of $\omega_z|\mathscr{R}$ has range $[\mathscr{R}'z]$. Writing $\mathscr{R}_0$ for $F\mathscr{R}F$ acting on $F(\mathscr{H})$ and ω_0 for $\omega|\mathscr{R}_0$, we have $\omega(A) = \omega_0(FAF)$, since $0 = \omega(I - E) \geqslant$ $\geqslant \omega(I - F) \geqslant 0$. As z is cyclic for $F(\mathscr{H})$ under $\mathscr{R}'F$, z is separating for $F\mathscr{R}F$.

Thus $\omega_0 = \omega_y|\mathscr{R}_0$, for some y in $F(\mathscr{H})$, and $\omega(A) = \omega_0(FAF) = (FAFy, y) = (Ay, y)$ for all A in $\mathscr{R}$.

3·6. *Lemma.* – If the normal state ω has the same support on the von Neumann algebra $\mathscr{R}$ as the state $\omega_z|\mathscr{R}$, where z is a unit cyclic vector for $\mathscr{R}$, then $\omega = \omega_x|\mathscr{R}$ for some unit cyclic vector x.

Proof. From the preceding Lemma, $\omega = \omega_y|\mathscr{R}$ for some unit vector y. Let E' be the projection (in $\mathscr{R}'$) with range $[\mathscr{R}y]$. The cyclic projection in $\mathscr{R}$ with z generating its range has the same central carrier as that in $\mathscr{R}'$ with z generating its range. Since the latter projection is I and the former is the support of $\omega_z|\mathscr{R}$, hence, of ω and $\omega_y|\mathscr{R}$, E' has central carrier I. Let $\omega_0(AE')$ be (Az, z) for each A in $\mathscr{R}$ (noting that $A \to AE'$ is an isomorphism of $\mathscr{R}$ onto $\mathscr{R}E'$). Then ω_0 is normal on $\mathscr{R}E'$, with the same support as $\omega_y|\mathscr{R}E'$, for $(Fz, z) = 0$ if and only if $(Fy, y) = (FE'y, y) = 0$. Thus $\omega_0 = \omega_v|\mathscr{R}E'$ for some unit vector v in $[\mathscr{R}y]$. Since $(Av, v) = (Az, z)$ for each A in $\mathscr{R}$, the mapping $Az \to Av$ of $\mathscr{R}z$ onto $\mathscr{R}v$ extends to a (partial) isometry V' of $\mathscr{H}$ onto $[\mathscr{R}v]$, with V' in $\mathscr{R}'$. Now $[\mathscr{R}v] \subseteq [\mathscr{R}y]$, so that $E' \sim I$ in $\mathscr{R}'$. Let W' be a partial isometry in $\mathscr{R}'$ with initial space $[\mathscr{R}y]$ and final space $\mathscr{H}$; and let x be $W'y$. Then $[\mathscr{R}x] = [\mathscr{R}W'y] = W'[\mathscr{R}y] = \mathscr{H}$, and $(Ax, x) = (AW'y, W'y) = (Ay, y) = \omega(A)$ for all A in $\mathscr{R}$.

3·7. *Theorem* (Unitary Implementation Theorem). – If φ is a *-isomorphism of the von Neumann algebra $\mathscr{R}_1$, with separating and cyclic vector x, onto the von Neumann algebra $\mathscr{R}_2$, with unit separating and cyclic vector y, then there is a unitary transformation U of the space $\mathscr{H}_1$, upon which $\mathscr{R}_1$ acts, onto the space $\mathscr{H}_2$, upon which $\mathscr{R}_2$ acts, such that $\varphi(A) = UAU^{-1}$ for each A in $\mathscr{R}_1$.

Proof. Let ω be the state $\omega_y \circ \varphi$ of $\mathscr{R}_1$. Since φ is a *-isomorphism of $\mathscr{R}_1$ onto $\mathscr{R}_2$, φ carries an orthogonal family of projections in $\mathscr{R}_1$ onto such a family in $\mathscr{R}_2$. Moreover, φ and φ^{-1} preserve order and, hence, the least upper bound (*i.e.* « sum ») of such a family. The complete additivity of ω follows from that of φ and $\omega_y|\mathscr{R}_2$. Since x is a separating vector for $\mathscr{R}_1$, $\omega = \omega_z|\mathscr{R}_1$ for some unit vector z in $\mathscr{H}_1$. Since y is separating for $\mathscr{R}_2$, ω is separating for $\mathscr{R}_1$, so that ω and $\omega_x|\mathscr{R}_1$ have the same support (in $\mathscr{R}_1$). From the preceding Lemma we can choose z so that $[R_1 z] = [\mathscr{R}_1 x] = \mathscr{H}_1$. Thus φ^{-1} is a cyclic representation of $\mathscr{R}_2$ as $\mathscr{R}_1$ on $\mathscr{H}_1$ with cyclic vector z such that $\omega_z \circ \varphi^{-1} = \omega_y|\mathscr{R}_2$, and y is a cyclic vector for $\mathscr{R}_2$ on $\mathscr{H}_2$. Hence there is a unitary transformation U of $\mathscr{H}_1$ onto $\mathscr{H}_2$ such that $\varphi(A) = UAU^{-1}$.

3·8. *Lemma.* – If ω is a normal state of the von Neumann algebra $\mathscr{R}$ acting on $\mathscr{H}$, and ω has the same support as $\omega_z|\mathscr{R}$, then there is a unit vector x such that $[\mathscr{R}x] = [\mathscr{R}z]$ and $\omega = \omega_x|\mathscr{R}$.

Proof. Let E' be the projection (in $\mathscr{R}'$) with range $[\mathscr{R}z]$, and define ω_0 on $\mathscr{R}E'$ by $\omega_0(AE') = \omega(A)$. Note that ω_0 is well defined, for if $AE' = 0$, then

$(A^*Az, z) = 0$; so that $\omega(A^*A)$, and, hence, $\omega(A)$, are 0. Since $C_{E'}$ is the central carrier of the projection E (in $\mathscr{R}$) with range $[\mathscr{R}' z]$, and E is the support of $\omega_z|\mathscr{R}$ and ω; we have that $\omega(A) = \omega(AC_{E'}) = \omega_0(AE')$. As the mapping $AC_{E'} \to AE'$ of $\mathscr{R}C_{E'}$ onto $\mathscr{R}E'$ is an isomorphism, ω_0 is a normal state of $\mathscr{R}E'$. Moreover, ω_0 has the same support as $\omega_z|\mathscr{R}E'$, for if $0 = (FE'z, z) = (Fz, z)$, then $\omega(F) = \omega_0(FE') = 0$. As z is cyclic for $E'(\mathscr{H})$ under $\mathscr{R}E'$, Lemma **3·6** applies, and there is a unit vector x in $E'(\mathscr{H})$ such that $\omega(A) = \omega_0(AE') = (AE'x, x) = (Ax, x)$, for all A in $\mathscr{R}$, and $[\mathscr{R}E'x] = [\mathscr{R}x] = E'(\mathscr{H}) = [\mathscr{R}z]$.

3·9. *Theorem.* – If $\mathscr{R}$ is a von Neumann algebra acting on the Hilbert space $\mathscr{H}$ and x, y are vectors in $\mathscr{H}$, then $[\mathscr{R}'x] \lesseqgtr [\mathscr{R}'y]$ if and only if $[\mathscr{R}x] \lesseqgtr [\mathscr{R}y]$.

Proof. If we have proved that $[\mathscr{R}'x] \sim [\mathscr{R}'y]$ entails $[\mathscr{R}x] \sim [\mathscr{R}y]$, and $[\mathscr{R}'x] \prec [\mathscr{R}'y]$, then $[\mathscr{R}'x] \sim [\mathscr{R}'y_0] < [\mathscr{R}'y]$, where $y_0 = Fy$ and F is the projection (in $\mathscr{R}$) with range $[\mathscr{R}'y_0]$. Then, by assumption, $[\mathscr{R}x] \sim [\mathscr{R}y_0] = [\mathscr{R}Fy] \leqslant [\mathscr{R}y]$;so that $[\mathscr{R}x] \leqslant [\mathscr{R}y]$. On the other hand, if $[\mathscr{R}x] \sim [Ry]$, then, by symmetry, $[\mathscr{R}'x] \sim [\mathscr{R}'y]$, contrary to the assumption $[\mathscr{R}'x] \prec [\mathscr{R}'y]$. Thus $[\mathscr{R}x] \prec [\mathscr{R}y]$.

It remains to establish that $[\mathscr{R}'x] \sim [\mathscr{R}'y]$ entails $[\mathscr{R}x] \sim [\mathscr{R}y]$. Suppose V in $\mathscr{R}$ is a partial isometry with initial space $[\mathscr{R}'x]$ and final space $[\mathscr{R}'y]$. Then $[\mathscr{R}'Vx] = V[\mathscr{R}'x] = [\mathscr{R}'y]$ and $[\mathscr{R}Vx] \subseteq [\mathscr{R}x] = [\mathscr{R}V^*Vx] \subseteq [\mathscr{R}Vx]$. Thus $[\mathscr{R}Vx] = [\mathscr{R}x]$. Replacing x by Vx, we may assume that $[\mathscr{R}'x] = [\mathscr{R}'y]$. In this case $\omega_x|\mathscr{R}$ and $\omega_y|\mathscr{R}$ have the same carrier, and we can find z in $[\mathscr{R}y]$ such that $\omega_z|\mathscr{R} = \omega_x|\mathscr{R}$ with $[\mathscr{R}z] = [\mathscr{R}y]$. The mapping $Ax \to Az$ extends to a partial isometry in $\mathscr{R}'$ with initial space $[\mathscr{R}x]$ and final space $[\mathscr{R}y]$. Thus $[\mathscr{R}x] \sim [\mathscr{R}y]$ (in $\mathscr{R}'$).

4. – A second approach to normal states.

Using the precise determination of extreme points on the unit ball of a C^*-algebra, a polar decomposition for normal linear functionals on a von Neumann algebra is established. With its aid, the Sakai-Radon-Nikodym theorem is proved. This is used to show that normal states are vector states in the presence of a separating vector.

4·1. *Theorem.* – The set of extreme points of the unit ball $\mathscr{A}_1$ of a C^*-algebra $\mathscr{A}$ consists precisely of those partially isometric operators V in $\mathscr{A}$ such that $(I - F)\mathscr{A}(I - E) = 0$, where $E = V^*V$ and $F = VV^*$.

Proof. Suppose, first, that V is an extreme point of $\mathscr{A}_1$, and let $\mathscr{A}(V^*V)$ be the C^*-subalgebra of $\mathscr{A}$ generated by V^*V. Then the spectrum $\sigma(V^*V)$ of V^*V is contained in $[0, 1]$. If a is a point of $(0, 1)$ and h is a continuous function, small on a small neighborhood of a, vanishing outside that neighborhood, and nonzero at a, then $\|V^*V(I \pm h(V^*V))^2\| \leq 1$. Thus $\|V(I \pm \pm h(V^*V))\| \leq 1$, and $V = \frac{1}{2}\big(V(I + h(V^*V)) + V(I - h(V^*V))\big)$. Since V is extreme on $\mathcal{A}_1$, $V = V + Vh(V^*V)$ and $0 = Vh(V^*V) = V^*Vh(V^*V)$. Since $h(a) \neq 0$, $a \notin \sigma(V^*V)$. Thus the spectrum of V^*V contains, at most, 0 and 1; so that V^*V is a projection E in $\mathcal{A}$. It follows that VV^* is a projection F in $\mathcal{A}$.

If A is an operator in the unit ball of $(I-F)\mathcal{A}(I-E)$ and z is a unit vector, then $z = x + y$ where $y = Ez$, $x = (I-E)z$, and

$$\|(V \pm A)z\|^2 = \|Vy \pm Ax\|^2 = \|FVy \pm (I-F)Ax\|^2 = \|Vy\|^2 + \|Ax\|^2 \leq 1\,.$$

Thus $V \pm A \in \mathcal{A}_1$. As $V = \frac{1}{2}(V + A + V - A)$, $V = V + A$ and $A = 0$. Hence $(I-F)\mathcal{A}(I-E) = (0)$.

Suppose U is a partially isometric operator in $\mathcal{A}$ with initial projection E and final projection F such that $(I-F)\mathcal{A}(I-E) = (0)$. If $U = \frac{1}{2}(A + B)$ with A, B in $\mathcal{A}_1$, then $1 = (Ux, Ux) = \frac{1}{2}\big((Ax, Ux) + (Bx, Ux)\big)$ if x is a unit vector in the range of E. As (Ax, Ux) and (Bx, Ux) lie in the unit disk in the complex numbers, and 1 is an extreme point of that disk, $1 = (Ax, Ux) = (Bx, Ux)$. From the limit case of the Schwarz inequality $Ax = Bx = Ux$. Thus $AE = BE = U$, and both A and B map the range of E isometrically onto that of F. We note that since both A and B have norm not exceeding 1, this last implies that $FA(I-E) = FB(I-E) = 0$. Otherwise, say, $FA(I-E) \neq 0$. There is an y in the range of $I-E$ such that $FAy = z$, for some unit vector z in the range of F. There is a unit vector x in the range of E such that $Ax = z$. Then

$$1 = \|z\| = \|FAy\| \leq \|y\| = a^{-1}\,;$$

$x\cos\theta + ya\sin\theta$ is a unit vector u for each θ in $[0, 2\pi]$. But $\|FAu\| = |\cos\theta + a\sin\theta| = (1+a^2)^{\frac{1}{2}}$, when $\operatorname{tg}\theta = a$, and $(1+a^2)^{\frac{1}{2}} > 1$.

Having noted that $FA(I-E) = FB(I-E) = 0$, and given $0 = (I-F)\cdot A(I-E) = (I-F)B(I-E)$, we have $A(I-E) = B(I-E) = 0$, so that $A = AE = U = BE = B$. Thus U is an extreme point of $\mathcal{A}_1$.

4·2. *Theorem* (Polar decomposition). – If $\mathcal{R}$ is a von Neumann algebra on $\mathcal{H}$ and ϱ is a linear functional weak-operator continuous on the unit ball $\mathcal{R}_1$ of $\mathcal{R}$, then there is a partial isometry U in $\mathcal{R}$, extreme on $\mathcal{R}_1$ such that ω is a positive normal linear functional on $\mathcal{R}$, where $\omega(A) = \varrho(UA)$ for each A in $\mathcal{R}$ and $\omega(U^*A) = \varrho(A)$.

Proof. Since ϱ is weak-operator continuous on $\mathcal{R}_1$, and $\mathcal{R}_1$ is weak-operator compact, there is some U in $\mathcal{R}_1$ such that $|\varrho(U)| = \|\varrho\|$. We may assume $\|\varrho\| = 1$; and, multiplying U by a suitable scalar of modulus 1, we may assume that $\varrho(U) = 1$. The subset of $\mathcal{R}_1$ at which ϱ takes the value 1 is compact and convex and its extreme points are extreme on $\mathscr{R}_1$. We may assume that U is an extreme point of $\mathscr{R}_1$, so that U is a partial isometry with initial projection E and final projection F such that $(I-F)\mathscr{R}(I-E)=(0)$. If $P=I-C_{I-E}$, then, since $I-E\leqslant C_{I-E}$, $P\leqslant E$. Since $C_{I-F}C_{I-E}=0$ and $I-F\leqslant C_{I-F}$, $I-P=C_{I-E}\leqslant I-C_{I-F}\leqslant F$, so that $U^*UP=P$ and $UU^*(I-P)=I-P$. Thus $UP+I-P(=V)$ and $U^*(I-P)+P(=W)$ are isometries ($V^*V=I$ and $W^*W=I$) in $\mathscr{R}$. Note that $W^*V=U$.

We may assume that $\mathscr{R}$ acting on $\mathscr{H}$ is the universal normal representation of $\mathscr{R}$. Since $A\to\varrho(W^*AV)$ assumes its norm, 1, at I, it is a (normal) state of $\mathscr{R}$, and there is a unit vector z in $\mathscr{H}$ such that $\varrho(W^*AV)=(Az,z)$ for all A in $\mathscr{R}$. Then $\varrho(A)=\varrho(W^*WAV^*V)=(AV^*z,W^*z)$ for all A in $\mathscr{R}$. As $1=\varrho(U)=(UV^*z,W^*z)$, and $\|V^*z\|\leqslant 1$, $\|W^*z\|\leqslant 1$, from the Cauchy-Schwarz Inequality, $\|u\|=\|v\|=1$, where $u=V^*z$ and $v=W^*z$. Let $\omega(A)$ be $\varrho(UA)$. Since ω takes its norm, 1, at I, ω is a (normal) state of $\mathscr{R}$. Since $1=\varrho(U)=\varrho(FU)=(Uu,Fv)$, we have $\|Fv\|=1=\|v\|$. Thus $Fv=v$. It follows that $\omega(U^*A)=\varrho(UU^*A)=\varrho(FA)=(FAu,v)=(Au,v)=\varrho(A)$, for all A in $\mathscr{R}$.

4·3. *Corollary.* – If ϱ is a normal linear functional of norm 1 on $\mathscr{R}$, then, in the universal normal representation of $\mathscr{R}$ (on $\mathscr{H}$) there are unit vectors u, v in $\mathscr{H}$ such that $\varrho=\omega_{u,v}|\mathscr{R}$.

4·4. *Lemma.* – If ϱ is a state of the C^*-algebra $\mathscr{A}$ and A in $\mathscr{A}$ is such that $B\to\varrho(BA)$ is Hermitian, then $|\varrho(HA)|\leqslant\|A\|\varrho(H)$ for each positive H in $\mathscr{A}$.

Proof. Since $B\to\varrho(BA)$ is Hermitian, $\varrho(B^*A)=\overline{\varrho(BA)}=\varrho(A^*B^*)$, for each B in $\mathscr{A}$. Thus $\varrho(BA^{2n})=\varrho(A^{*n}BA^n)=\overline{\varrho(A^{*n}B^*A^n)}=\overline{\varrho(B^*A^{2n})}$. With H positive in $\mathscr{A}$

$$|\varrho(HA)|=|\varrho(H^{\frac12}H^{\frac12}A)|\leqslant\varrho(A^*HA)^{\frac12}\varrho(H)^{\frac12}=\varrho(HA^2)^{\frac12}\varrho(H)^{\frac12},$$

and $|\varrho(HA^{2^n})|\leqslant\varrho(HA^{2^{n+1}})^{\frac12}\varrho(H)^{\frac12}$. Hence

$$|\varrho(HA)|\leqslant\varrho(HA^4)^{\frac14}\varrho(H)^{\frac12}\varrho(H)^{\frac14}\leqslant\dots\leqslant\varrho(HA^{2^n})^{2^{-n}}\varrho(H)^{\frac12+\dots+1/2^n}\leqslant$$

$$\leqslant(\|\varrho\|\,\|H\|)^{2^{-n}}\|A\|\,\varrho(H)^{1-1/2^n}$$

and

$$|\varrho(HA)|\leqslant\|A\|\,\varrho(H)\,.$$

Remark. If $B\to\varrho(AB)$ is Hermitian,

$$|\varrho(AH)|=|\varrho(HA^*)|\leqslant\|A^*\|\,\varrho(H)=\|A\|\,\varrho(H)\,.$$

4·5. *Lemma.* – If $\mathscr{A}$ is a self-adjoint algebra of operators on the Hilbert space $\mathscr{H}$ and ω is a positive linear functional on $\mathscr{A}$ such that $\omega\leqslant\omega_x|\mathscr{A}$ for some vector x in $\mathscr{H}$, then there is a positive operator T' in the unit ball $\mathscr{A}'_1$ of $\mathscr{A}'$ such that $\omega(A) = \omega_x(T'A) = \omega_{T'^{\frac{1}{2}}x}(A)$ for all A in $\mathscr{A}$.

Proof. With $\varphi(Ax, Bx)$ defined as $\omega(B^* A)$

$$|\varphi(Ax, Bx)|^2 = |\omega(B^* A)|^2 \leqslant \omega(A^* A)\,\omega(B^* B) \leqslant \|Ax\|^2 \|Bx\|^2 .$$

Thus φ is well-defined and is a Hermitian, bilinear form on $\mathscr{A}x$ bounded by 1. Hence φ has a unique extension to $[\mathscr{A}x]$, and there is a positive T' of norm not exceeding 1 such that $(T'Ax, Bx) = \varphi(Ax, Bx) = \omega(B^*A)$. Thus $\omega(A) = (T'Ax, x) = \omega_x(T'A)$ for all A in $\mathscr{A}$. Extending T' by defining it to be 0 on $\mathscr{H} - [\mathscr{A}x]$ leaves it positive, of norm not exceeding 1, and leaves the foregoing equalities unaltered. Since

$$(T'ABx, Cx) = \omega(C^* AB) = (T'Bx, A^* Cx) = (AT'Bx, Cx)$$

for all A, B, C in $\mathscr{A}$, $T'A - AT'$ is 0 on $[\mathscr{A}x]$. As T' is 0 on $\mathscr{H} - [\mathscr{A}x]$, $T'A - AT' = 0$ for all A in $\mathscr{A}$, and $T' \in \mathscr{A}'_1$. Thus $\omega(A) = (T'Ax, x) = (AT'^{\frac{1}{2}}x, T'^{\frac{1}{2}}x) = \omega_{T'^{\frac{1}{2}}x}(A)$ for all A in $\mathscr{A}$.

4·6. *Theorem* (Sakai-Radon-Nikodym). – If ω and ω_0 are normal, positive, linear functionals on a von Neumann algebra $\mathscr{R}$ such that $\omega_0 \leqslant \omega$, then there is a positive operator T_0 in the unit ball of $\mathscr{R}$ such that $\omega_0(A) = \omega(T_0 A T_0)$ for all A in $\mathscr{R}$.

Proof. Working with $\mathscr{R}$ in its universal normal representation, $\omega = \omega_x|\mathscr{R}$. The support E of $\omega_x|\mathscr{R}$ has range $[\mathscr{R}'x]$, and ω_0 has support dominated by E. Dealing with $E\mathscr{R}E$ acting on $[\mathscr{R}'x]$, we may assume that $\omega = \omega_x|\mathscr{R}$ and $[\mathscr{R}'x] = \mathscr{H}$, the Hilbert space on which $\mathscr{R}$ acts. From Lemma 4·5 there is a positive operator T'_0 in the unit ball of $\mathscr{R}'$ such that $\omega = \omega_{T'_0x}|\mathscr{R}$. From Theorem 4·2 (polar decomposition of normal functionals) there is a partial isometry V' in $\mathscr{R}'$ such that $\omega_{x,V'^*T'_0x}|\mathscr{R}'$ is positive and $\omega_{x,T'_0x}|\mathscr{R}' = \omega_{x,V'V'^*T'_0x}|\mathscr{R}'$. Since $[\mathscr{R}'x] = \mathscr{H}$, $T'_0x = V'V'^*T'_0x$. From Lemma 4·4, with H' a positive operator in $\mathscr{R}'$, $\omega_{x,V'^*T'_0x}(H') \leqslant \|T'_0V'\|\,\omega_x(H') \leqslant \omega_x(H')$. From Lemma 4·5 there is a positive operator $T_0^{\frac{1}{2}}$ in the unit ball of $\mathscr{R}$ such that $\omega_{x,V'^*T'_0x}|\mathscr{R}' = \omega_{T_0^{\frac{1}{2}}x}|\mathscr{R}'$. Hence $(A'x, V'^*T'_0x) = (A'x, T_0x)$ for all A' in $\mathscr{R}'$, and $V'^*T'_0x = T_0x$. Since $T'_0x = V'V'^*T'_0x$, $\omega_{T_0x}|\mathscr{R} = \omega_{T'_0x}|\mathscr{R} = \omega$.

4·7. *Lemma.* – If $\mathscr{R}$ is a von Neumann algebra acting on the Hilbert space $\mathscr{H}$ and H is a positive operator in $\mathscr{R}$, then $[\mathscr{R}Hx_0]$ contains x_0 if $Nx_0 = 0$, where N is the projection on the null space of H.

Proof. We may assume that $\mathscr{R}$ is the (Abelian) von Neumann algebra generated by H (and N, and I). Then $\mathscr{R} \simeq C(X)$ with X extremely disconnected. Let $E_{1/n}$ be the (spectral) projection (for H) in $\mathscr{R}$ corresponding to the largest clopen set on which the function, f, representing H is not greater than $1/n$.

Then $E_{1/n}$ tends strongly to E_0 as $n \to \infty$, and $HE_0 = 0$. On the other hand, since $HN = 0$, f is 0 on the clopen set corresponding to N, so that $N \leqslant E_{1/n}$ for all n. Thus $N = E_0$ and $E_{1/n}x_0 \to E_0 x_0 = Nx_0 = 0$. If $F_n = I - E_{1/n}$, $F_n x_0 \to x_0$.

If K_n is the operator in $\mathcal{R}$ corresponding to the function which is $1/f$ on the clopen set corresponding to F_n, then $HK_n = F_n$. Thus $K_n Hx_0 \to x_0$ and $[\mathcal{R}Hx_0]$ contains x_0.

4.8. *Theorem.* – If ω is a normal state of the von Neumann algebra $\mathcal{R}$ acting on the Hilbert space $\mathcal{H}$ and z is a unit separating vector for $\mathcal{R}$, then $\omega = \omega_x|\mathcal{R}$, for some unit vector x.

Proof. If ω_0 is $\omega + \omega_z|\mathcal{R}$, then ω_0 is separating for $\mathcal{R}$ and the representation φ engendered by $\frac{1}{2}\omega_0$ is a faithful mapping of $\mathcal{R}$ onto the von Neumann algebra $\varphi(\mathcal{R})$ acting on $\mathcal{K}$. Let y_0 be a cyclic vector for $\varphi(\mathcal{R})$ in $\mathcal{K}$ such that $(\varphi(A)y_0, y_0) = \omega_0(A)$, for all A in $\mathcal{R}$. As $\omega \leqslant \omega_0$ and $\omega_z \leqslant \omega_0$, there is a unit vector x_0 in $\mathcal{K}$ such that $\omega(A) = (\varphi(A)x_0, x_0)$, and an operator $\varphi(H)$ in $\varphi(\mathcal{R})$ such that $\omega_z(A) = (\varphi(A)\varphi(H)y_0, \varphi(H)y_0)$ for all A in $\mathcal{R}$, and $0 \leqslant H \leqslant I$. If N is the projection on the null space of H, $\varphi(N)$ is the projection on the null space of $\varphi(H)$, and $0 = (\varphi(N)\varphi(H)y_0, \varphi(H)y_0) = \omega_z(N)$. Thus N and $\varphi(N)$ are 0. From the preceding Lemma, $[\varphi(\mathcal{R})\varphi(H)y_0]$ contains y_0 and coincides with $\mathcal{K}$. The mapping $Az \to \varphi(A)\varphi(H)y_0$ extends to a unitary transformation U of $[\mathcal{R}z]$ onto $\mathcal{K}$ such that $UAU^{-1} = \varphi(A)$ for all A in $\mathcal{R}$. If x is $U^{-1}x_0$, then x is a unit vector, and $(Ax, x) = (UAU^{-1}x_0, x_0) = (\varphi(A)x_0, x_0) = \omega(A)$ for all A in $\mathcal{R}$.

4.9. *Lemma.* – If $\mathcal{R}$ is a von Neumann algebra acting on $\mathcal{H}$ and $\omega_{x,y}|\mathcal{R}$ is a state of $\mathcal{R}$, then there is a unit vector z in $\mathcal{H}$ such that $\omega_z|\mathcal{R} = \omega_{x,y}|\mathcal{R}$.

Proof. If H is a positive operator in $\mathcal{R}$, then $0 \leqslant \omega_{x-y}(H) = (Hx, x) + (Hy, y) - (Hx, y) - (Hy, x)$. As $\omega_{x,y}|\mathcal{R}$ is a state, $(Hx, y) = (y, Hx) = (Hy, x)$, and $2(Hx, y) \leqslant (Hx, x) + (Hy, y)$. Thus $4\omega_{x,y}(H) \leqslant \omega_{x+y}(H)$. From Lemma 4.5, $\omega_{x,y}|\mathcal{R} = \omega_z|\mathcal{R}$ for some unit vector z.

Employing this Lemma in conjunction with the last paragraph of the proof of Theorem 4.2, we have:

4.10. *Corollary.* – If $\|\omega_{x-y}|\mathcal{R}\| = 1$, for some vectors x, y in the Hilbert space $\mathcal{H}$ on which the von Neumann algebra $\mathcal{R}$ acts, then there are unit vectors u, v in $\mathcal{H}$ such that $\omega_{x,y}|\mathcal{R} = \omega_{u,v}|\mathcal{R}$.

5. – The type of the commutant.

In this Section the special properties of the trace re-enter the theory for the purpose of establishing that the commutants of Type I, II or III von Neumann algebras are themselves of Type I, II or III, respectively.

5˙1. *Lemma.* – If x_0 is a cyclic unit trace vector for the von Neumann algebra $\mathcal{R}$ acting on $\mathcal{H}$ and A' is a self-adjoint operator in $\mathcal{R}'$, there are self-adjoint operators A_n in $\mathcal{R}$ such that $A_n x_0 \to A' x_0$.

Proof. Since x_0 is cyclic for $\mathcal{R}$, there are operators B_n in $\mathcal{R}$ such that $B_n x_0 \to \to A' x_0$. If B is in $\mathcal{R}$, then

$$(B_n^* x_0, Bx_0) = (x_0, B_n B x_0) = (x_0, B B_n x_0) \to (x_0, BA' x_0) = (A' x_0, Bx_0).$$

Moreover

$$\|(B_n^* - B_m^*) x_0\|^2 = \big((B_n - B_m)(B_n^* - B_m^*) x_0, x_0\big) =$$
$$= \big((B_n^* - B_m^*)(B_n - B_m) x_0, x_0\big) = \|(B_n - B_m) x_0\|^2 \to 0 \text{ as } n, m \to \infty.$$

Thus $(B_n^* x_0)$ converges to some vector y. From the first computation, $(y, Bx_0) = (A' x_0, Bx_0)$ for each B in $\mathcal{R}$. As $[\mathcal{R} x_0] = \mathcal{H}$, $B_n^* x_0 \to y = A' x_0$. Hence $A_n x_0 \to A' x_0$ where $A_n = \frac{1}{2}(B_n + B_n^*)$.

5˙2. *Theorem.* – If the von Neumann algebra $\mathcal{R}$ acting on the Hilbert space $\mathcal{H}$ is finite and has a separating vector and a cyclic vector, then there is a cyclic trace vector for both $\mathcal{R}$ and $\mathcal{R}'$.

Proof. Let Tr be the (centre-valued) trace on $\mathcal{R}$ and y_0 a separating vector for $\mathcal{R}$. If $\omega(A) = \big(\mathrm{Tr}(A) y_0, y_0\big)$ for all A in $\mathcal{R}$, then ω is a normal state of $\mathcal{R}$. Since y_0 is separating for $\mathcal{R}$, ω is a vector state $\omega_{x_0}|\mathcal{R}$ of $\mathcal{R}$, and ω is separating for $\mathcal{R}$. If z_0 is cyclic for $\mathcal{R}$; $[\mathcal{R} z_0] = \mathcal{H} \leqslant [\mathcal{R} x_0]$ since $[\mathcal{R}' z_0] \leqslant [\mathcal{R}' x_0] = \mathcal{H}$. Thus $[\mathcal{R} x_0] \sim \mathcal{H}$, and there is a partial isometry V' in $\mathcal{R}'$ with initial space $[\mathcal{R} x_0]$ and final space $\mathcal{H}$. Then $\omega_{V' x_0}|\mathcal{R} = \omega_{x_0}|\mathcal{R}$ and $[\mathcal{R} V' x_0] = V'[\mathcal{R} x_0] = \mathcal{H}$. We may assume that x_0 is cyclic for $\mathcal{R}$. As $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, $\omega(AB) = \omega(BA)$, and x_0 is a (cyclic) trace vector for $\mathcal{R}$.

We complete the proof by showing that a cyclic trace vector x_0 for a von Neumann algebra $\mathcal{R}$ is a (cyclic) trace vector for $\mathcal{R}'$. With A', B' self-adjoint operators in $\mathcal{R}'$, applying Lemma **5˙1**, we can find sequences (A_n) (B_n) of self-adjoint operators in $\mathcal{R}$ such that $A_n x_0 \to A' x_0$, $B_n x_0 \to B' x_0$. Then $(A_n x_0, B_n x_0) \to \to (A' x_0, B' x_0)$ and $(B_n x_0, A_n x_0) \to (B' x_0, A' x_0)$. But

$$(A_n x_0, B_n x_0) = (B_n A_n x_0, x_0) = (A_n B_n x_0, x_0) = (B_n x_0, A_n x_0).$$

Thus

$$(B' A' x_0, x_0) = (A' x_0, B' x_0) = (B' x_0, A' x_0) = (A' B' x_0, x_0).$$

The same, now, holds for all A', B' in $\mathcal{R}'$, and x_0 is a trace vector for $\mathcal{R}'$ (cyclic since it is separating for $\mathcal{R}$).

5.3. *Theorem.* – If the von Neumann algebra $\mathscr{R}$ acting on the Hilbert space $\mathscr{H}$ has a cyclic trace vector x_0, then $\mathscr{R}'$ is finite. For each A in $\mathscr{R}$ there is a unique A' in $\mathscr{R}'$ such that $Ax_0 = A'x_0$. The mapping $A \to A'$ is a *anti-isomorphism of $\mathscr{R}$ onto $\mathscr{R}'$.

Proof. From the preceding theorem, x_0 is a cyclic trace vector for $\mathscr{R}'$ so that $\mathscr{R}'$ is finite. If U is a unitary operator in $\mathscr{R}$, $\omega_{Ux_0}|\mathscr{R} = \omega_{x_0}|\mathscr{R}$, so that the mapping $Tx_0 \to TUx_0$ extends to a unitary operator U' in $\mathscr{R}'$ (noting that $[\mathscr{R}x_0] = [\mathscr{R}Ux_0] = \mathscr{H}$), and $Ux_0 = U'x_0$. Since each operator A in $\mathscr{R}$ is a linear combination of (at most, four) unitary operators in $\mathscr{R}$, there is an operator A' in $\mathscr{R}'$ such that $Ax_0 = A'x_0$. As x_0 is separating for $\mathscr{R}'$, there is just one such A'. Now $(aA + B)x_0 = aA'x_0 + B'x_0 = (aA' + B')x_0$ so that $A \to A'$ is linear. Moreover, $ABx_0 = AB'x_0 = B'Ax_0 = B'A'x_0$, whence $(AB)' = B'A'$. By symmetry $A \to A'$ has a two-sided inverse, so that it is an anti-isomorphism of $\mathscr{R}$ onto $\mathscr{R}'$. Noting, again, that, with H and K self-adjoint operators in $\mathscr{R}$, $(Hx_0, Kx_0) = (KHx_0, x_0) = (HKx_0, x_0) = (Kx_0, Hx_0) = \overline{(Hx_0, Kx_0)}$, we see that (Hx_0, Kx_0) is real. Thus, if A is a self-adjoint operator in $\mathscr{R}$ and B is an arbitrary operator in $\mathscr{R}$, $(A'Bx_0, Bx_0) = (A'x_0, B^*Bx_0) = (Ax_0, B^*Bx_0)$, which is real. Thus A' is self-adjoint, and $A \to A'$ is a *-anti-isomorphism of $\mathscr{R}$ onto $\mathscr{R}'$.

Remark C. In the preceding argument we established that for each A in $\mathscr{R}$ there is a unique A' in $\mathscr{R}'$ such that $Ax_0 = A'x_0$ by appealing to the decomposition of operators in $\mathscr{R}$ as linear combinations of unitary operators in $\mathscr{R}$. Other routes to this conclusion are available. With H and A positive operators in $\mathscr{R}$

$$0 \leqslant \omega_{Ax_0}(H) = (HAx_0, Ax_0) = (AHAx_0, x_0) = (HA^2x_0, x_0) = (H^{\frac{1}{2}}A^2H^{\frac{1}{2}}x_0, x_0) \leqslant \|A^2\|(Hx_0, x_0) .$$

From Dye's lemma, there is a positive operator A' in $\mathscr{R}$ such that $\omega_{A'x_0}|\mathscr{R} = \omega_{Ax_0}|\mathscr{R}$. It follows that

$$(TA'x_0, A'x_0) = (Tx_0, A'^2x_0) = (TAx_0, Ax_0) = (A^2Tx_0, x_0) = (Tx_0, A^2x_0) .$$

Since x_0 is cyclic for $\mathscr{R}$, $A'^2x_0 = A^2x_0$. Decomposing a self-adjoint operator as the difference of two positive operators, and each operator as the sum of a self-adjoint and skew-adjoint operator, we have the desired mapping ($A \to A'$ of Theorem **5.2**). In this approach the mapping is seen to be adjoint preserving.

Remark D. If less emphasis on the use of the trace vector is desired, the Sakai-Radon-Nikodym theorem can be used. With H' positive in $\mathscr{R}'$ and A positive in $\mathscr{R}$, $\omega_{Ax_0}(H') = (H'A^2x_0, x_0) \leqslant \|A^2\|(H'x_0, x_0)$, so that there is a positive A' in $\mathscr{R}'$ such that $\omega_{Ax_0}|\mathscr{R}' = \omega_{A'x_0}|\mathscr{R}'$. Thus $(T'x_0, A^2x_0) = (T'A'x_0, A'x_0) = (T'x_0, A'^2x_0)$. As x_0 is cyclic for $\mathscr{R}'$, $A^2x_0 = A'^2x_0$, and we proceed as before.

5·4. *Corollary.* – If $\mathscr{A}$ is an Abelian von Neumann algebra acting on the Hilbert space $\mathscr{H}$ and x_0 is cyclic for $\mathscr{A}$ then $\mathscr{A} = \mathscr{A}'$ (*i.e.* $\mathscr{A}$ is maximal Abelian).

Proof. Since $\mathscr{A}$ is Abelian, $\mathscr{A} \subseteq \mathscr{A}'$, and x_0 is cyclic for $\mathscr{A}'$. Thus x_0 is separating for $\mathscr{A}$, and x_0 is a trace vector for $\mathscr{A}$. Hence $\mathscr{A}$ and $\mathscr{A}'$ are anti-isomorphic, so that $\mathscr{A}'$ is Abelian. Thus $\mathscr{A}' \subseteq \mathscr{A}'' = \mathscr{A}$; and $\mathscr{A} = \mathscr{A}'$.

5·5. *Corollary.* – If the projection E' with range $[\mathscr{R}x]$ in the von Neumann algebra $\mathscr{R}'$ acting on $\mathscr{H}$ is Abelian, then the projection E with range $[\mathscr{R}'x]$ is Abelian in $\mathscr{R}$.

Proof. As the commutant, $E'\mathscr{R}'E'$, of $\mathscr{R}E'$ acting on $E'(\mathscr{H})$ is Abelian, $E'\mathscr{R}'E'$ is the center of $\mathscr{R}E'$. Since x is cyclic for $E'(\mathscr{H})$ under $\mathscr{R}E'$, EE' with range $[E'\mathscr{R}'E'x]$ has central support E' in $\mathscr{R}E'$. Thus $E'\mathscr{R}'E'$ is isomorphic to $E'\mathscr{R}'E'E$ (which is, accordingly, Abelian), and $E'\mathscr{R}'E'E$ has commutant $E'E\mathscr{R}EE'$ on $EE'(\mathscr{H})$. Now, x is cyclic for $EE'(\mathscr{H})$ under $E'\mathscr{R}'E'E$; so that, from the preceding corollary, $E'\mathscr{R}'E'E$ is maximal Abelian. Thus $E'E\mathscr{R}EE'$ is Abelian. But $E'E\mathscr{R}EE' = E\mathscr{R}EE'$ and $E\mathscr{R}EE'$ is isomorphic to $E\mathscr{R}E$. Thus E is Abelian in $\mathscr{R}$.

5·6. *Corollary.* – If $\mathscr{R}$ is a finite von Neumann algebra acting on $\mathscr{H}$ and x is a vector in $\mathscr{H}$ cyclic for the projection E' in $\mathscr{R}'$, then E' is finite in $\mathscr{R}'$.

Proof. The von Neumann algebra $\mathscr{R}E'$ acting on $E'(\mathscr{H})$ is finite with cyclic vector x and commutant $E'\mathscr{R}'E'$. If E is the projection in $\mathscr{R}E'$ with range $[E'\mathscr{R}'E'x]$, then $E\mathscr{R}E'E (= E\mathscr{R}E)$ is finite with cyclic and separating vector x on $E(\mathscr{H})$ and with commutant $E'\mathscr{R}'E'E$. As x is cyclic for $\mathscr{R}E'$ in $E'(\mathscr{H})$, E has central carrier E' in $\mathscr{R}E'$, and $E'\mathscr{R}'E'E$ is isomorphic to $E'\mathscr{R}'E'$. But, from the preceding theorems $E\mathscr{R}E$ is anti-isomorphic to $E'\mathscr{R}'E'E$. Thus $E'\mathscr{R}'E'E$ and $E'\mathscr{R}'E'$ are finite, so that E' is finite in $\mathscr{R}'$.

5·7. *Corollary.* – If $\mathscr{R}$ is a von Neumann algebra of Type II acting on the Hilbert space $\mathscr{H}$, then $\mathscr{R}'$ is of Type II.

Proof. Let E be a finite projection in $\mathscr{R}$ with central carrier I. Then $E\mathscr{R}E$ is finite and $\mathscr{R}'E$ is isomorphic to $\mathscr{R}'$. We may assume $\mathscr{R}$ is of Type II_1. From the preceding corollary, each cyclic projection in $\mathscr{R}'$ is finite. As each projection in $\mathscr{R}'$ is a sum of cyclic projections, $\mathscr{R}'$ contains no central portion of Type III.

If $\mathscr{R}'$ contains an Abelian projection it contains a cyclic Abelian projection. In this case, from Corollary **5·5**, $\mathscr{R}$ contains a cyclic Abelian projection—contradicting the assumption that $\mathscr{R}$ is of Type II. Thus $\mathscr{R}'$ is of Type II.

5·8. *Corollary.* – If $\mathscr{R}$ is of Type I or of Type III, then $\mathscr{R}'$ is of Type I or of Type III, respectively.

Proof. If $\mathscr{R}$ is of Type I it contains a family of cyclic Abelian projections with central carriers having sum I. From Corollary **5·5**, $\mathscr{R}'$ contains a family of Abelian projections with the same central carriers. Thus $\mathscr{R}'$ is of Type I.

If $\mathscr{R}$ is of Type III, $\mathscr{R}'$ has no central portion of Type I, from the foregoing, nor any of Type II from the preceding corollary. Thus $\mathscr{R}'$ is of Type III.